Partial Ranking by Incomplete Pairwise Comparisons Using Preference Subsets

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Abstract. In multi-criteria decision making the decision maker need to assign weights to criteria for evaluation of alternatives, but decision makers usually find it difficult to assign precise weights to several criteria. On the other hand, decision makers may readily provide a number of preferences regarding the relative importance between two disjoint subsets of criteria. We extend a procedure by L. V. Utkin for ranking alternatives based on decision makers' preferences. With this new method we may evaluate and rank partial sequences of preferences between two subsets of criteria. To achieve this ranking it is necessary to model the information value of an incomplete sequence of preferences and compare this with the belief-plausibility of that sequence in order to find the partial ranking of preferences with maximum utility.

Keywords: belief function, Dempster-Shafer theory, preferences, multi-criteria decision making, pairwise comparison, ranking.

1 Introduction

In multi-criteria decision making (MCDM) decision makers needs to evaluate and rank different alternatives using several criteria (e.g., measures of effectiveness; MOEs). To be able to rank the alternatives they usually seek a weighting of these criteria, but weights may be unavailable and decision makers may find it impossible to assign precise weights to all criteria. An initial step can be to filter all alternatives under consideration by Pareto filtering [6, 16]. This will eliminate all alternatives that can never be selected regardless of which weight assignment is adopted for the criteria. This will reduce the problem size, but the same problem with assigning weights remains. However, it is often possible for decision makers to express an order of importance between all different criteria, or at least to express a preference between two different

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subsets of all criteria.

In this paper we develop an extension to a procedure by Utkin [20] for ranking alternatives based on multiple decision makers' preferences in MCDM. We let a group of decision makers express any number of preferences regarding the relative importance between any two disjoint subsets of criteria. We derive a method for finding a partial preference order of all measures of effectiveness. This method will accept any preference expression about the MOEs from multiple decision makers. For example, expressions such as "*measure of effectiveness MOE_i is more important than measure of effectiveness MOE_j*"; $MOE_i \succcurlyeq MOE_j$, or expressions regarding two different subsets of all measures such as "*measures MOE_i and MOE_j are more important than measures MOE_k and MOE_l*"; $\{MOE_i, MOE_j\} \succcurlyeq \{MOE_k, MOE_l\}$. As we extend the preference assignment approach developed by Utkin we combine it with a preference ranking approach by Schubert [12] to derive a partial ranking of all MOEs. When the best sequence of preferences (of measures of effectiveness) is found we can weight all alternatives and select the best alternative with the highest value. This alternative can be further analysed to explain the cause of success [14].

Another approach is provided by Masson and Denœux [11] that extends a methodology by Tritchler and Lockwood [19]. In [19] simple support functions regarding each singleton pair of preferences $m^{\Theta_{ij}}$ are assigned on individual frames Θ_{ij} by experts. After all assignments are extended to a common frame of discernment and combined the most plausible linear ordering of all preference is found. In [11] a linear programming approach is proposed to solve the problem in an efficient way. The methodology is further extended to some partial rankings of preferences where a hierarchical clustering approach selects which partial orders of preferences are evaluated based on plausibility. The final choice of preferred partial order is left to the user.

In Sec. 2 we assign basic belief masses based on all decision makers' pairwise preferences of any two subsets of all measures of effectiveness. In Sec. 3 we calculate a decision maker's belief and plausibility in partial sequences of preferences. In Sec. 4 we derive the information value of a partial sequence of preferences. Based on the results of the previous two sections we calculate the utility of each partial sequence of preferences as a product of two functions corresponding, on the one hand, to belief and plausibility in the proposition and, on the other hand, the information value of the proposition (Sec. 4). Finally, conclusions are drawn (Sec. 5).

2 Assignment of Decision Makers' Preferences

We will keep track of all preferences expressed by all decision makers. This includes both preferences about the order of importance among single measures and among subsets of measures. For each expression we count the number of decision makers giving the same preferences and sum-up the total number of assigned preferences by all decision makers

$$c_{AB}(\{MOE_i\}_{i \in A} \succcurlyeq \{MOE_j\}_{j \in B}), \quad (1)$$

where $\emptyset \neq A, B \subseteq \{i\}_{i=1}^{|\{MOE_i\}_i|} = I$, i.e., A and B are subsets of and index set I of indices corresponding to the set of all MOEs. Any number of these c_{AB} may be equal to zero due to a lack of assigned preferences regarding some subsets of MOEs.

The preferences assigned between two subsets of measures can be simplified to a set of preferences among single pairs of measures [20]. We have,

$$\{MOE_i\}_{i \in A} \succcurlyeq \{MOE_j\}_{j \in B} = \{MOE_i \succcurlyeq MOE_j\}_{i \in A, j \in B} . \tag{2}$$

From the counts of assigned preferences (1) we derive a basic belief assignment within belief function theory [3, 4, 17]. In this setting of our problem representation, the frame of discernment (i.e., the set of all possible preference rankings) is

$$\Theta = 2^{\{MOE_i \succcurlyeq MOE_j\}_{i,j \in I}} . \tag{3}$$

However, only a subset of Θ corresponding to chains of preferences will be under investigation in this approach (see (8) in Sec. 3).

We have the following basic belief assignment, using (1),

$$m_{AB}(\{MOE_i \succcurlyeq MOE_j\}_{i \in A, j \in B}) = \frac{1}{N} c_{AB}(\{MOE_i \succcurlyeq MOE_j\}_{i \in A, j \in B}) \tag{4}$$

where N is the total sum of all counts

$$N = \sum_{AB} c_{AB}(\{MOE_i \succcurlyeq MOE_j\}_{i \in A, j \in B}) . \tag{5}$$

While it is possible to change the representation in (4) and (5) using (2), it is not possible to divide the basic belief mass among the different preferences in $\{MOE_i \succcurlyeq MOE_j\}_{i \in A, j \in B}$ as we have no information on how to divide it among the different preferences. Instead the entire mass must remain on the whole set.

3 A Decision Maker's Belief in Preferences

From the basic belief assignments (4) we may calculate belief and plausibility for any element of the frame of discernment.

While it is possible to calculate belief and plausibility in each single measures of performance such as,

$$\{MOE_j\} \succcurlyeq \{MOE_i\}_{i \in I'} \forall j \tag{6}$$

where $|\{MOE_j\}| = 1$ (as was done in [15]) we will instead calculate belief and plausibility in incomplete rankings of all measures. Utkin [20] considered complete rankings B of all measure as an alternative approach to calculating belief and plausibility in (6) where plausibility was calculated for a sequence of preferences

$$B_{i_1 i_n} = \{(MOE_{i_1} \succcurlyeq MOE_{i_2}) \cap (MOE_{i_2} \succcurlyeq MOE_{i_3}) \cap \ldots \cap (MOE_{i_{n-1}} \succcurlyeq MOE_{i_n})\} , \tag{7}$$

containing all preferences once. Here belief in any complete sequence is zero as we only have basic belief assignments in sets of preference relations (4) that are all proper supersets to (7) even in the case when the supported set in (4) is a singleton set, as (7) is a sequence of intersections (*not* unions).

We will derive an extension of (7) where we allow any sequence that is an intersection of subsets of all preference relations, including but not limited to singleton sets,

$$B^*_{i_1 i_n} = \{(B_{i_1} \succcurlyeq B_{i_2}) \cap (B_{i_2} \succcurlyeq B_{i_3}) \cap \ldots \cap (B_{i_{n-1}} \succcurlyeq B_{i_n})\}, \tag{8}$$

where B_{i_j} is a subset of all measures $\{MOE_i\}_{i \in A}$ such that the intersection $B_{i_j} \cap B_{i_k} = \emptyset$ and $\cup B_{i_j} = \{MOE_i\}_{i \in I'}$ $\forall j$.

Using the representation of (8) we have focal elements $m(B_{i_j} \succcurlyeq B_{i_k})$ for many (but *not* necessarily all) indices i_j, i_k, and may calculate belief and plausibility in $B^*_{i_1 i_n}$. We get beliefs,

$$\mathrm{Bel}(B^*_{i_1 i_2}) = m(B_{i_1} \succcurlyeq B_{i_2}), \quad n = 2, \tag{9}$$

$$\mathrm{Bel}(B^*_{i_1 i_n}) = 0, \qquad n \geq 3 \tag{10}$$

where belief in any nonsingleton preferences is always zero (as mentioned above), and may in addition calculate plausibility in any partial sequence of preference as

$$\mathrm{Pls}(B^*_{i_1 i_n}) = \sum_{(B_{i_j} \succcurlyeq B_{i_k}) \in \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} \,\middle|\, \cap \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} = B^*_{i_1 i_n}} m(B_{i_j} \succcurlyeq B_{i_k}) \tag{11}$$

where the sum is taken over all focal elements $(B_{i_j} \succcurlyeq B_{i_k})$, $1 \leq j$, $k = j+1 \leq n$, that are included in $B^*_{i_1 i_n}$.

Given the calculated belief and plausibility we may compare all partial sequences of preferences $B^*_{i_1 i_n}$. If the belief intervals of two different sequences of partial preferences are not overlapping then clearly the higher believed sequence is more preferred.

When an interval $[\mathrm{Bel}(B^*_{i_j i_k}), \mathrm{Pls}(B^*_{i_j i_k})]$ is fully included in an interval $[\mathrm{Bel}(B^*_{i_l i_m}), \mathrm{Pls}(B^*_{i_l i_m})]$ it is not immediately clear which is the preferred partial sequence of preferences; $B^*_{i_j i_k}$ or $B^*_{i_l i_m}$. We can interpolate with a parameter $\rho \in [0, 1]$ in each belief-plausibility interval in order to find the preferred partial sequence of preferences [12]. However, we have no information regarding the value of ρ, and any assumption about ρ will be unwarranted.

Instead we may calculate the point ρ_{jklm} where the two partial sequences of preferences $B^*_{i_j i_k}$ and $B^*_{i_l i_m}$ are equally preferred. When

$$[\mathrm{Bel}(B^*_{i_l i_m}), \mathrm{Pls}(B^*_{i_l i_m})] \supset [\mathrm{Bel}(B^*_{i_j i_k}), \mathrm{Pls}(B^*_{i_j i_k})] \tag{12}$$

we have

$$\rho_{jklm} = \frac{\mathrm{Bel}(B^*_{i_j i_k}) - \mathrm{Bel}(B^*_{i_l i_m})}{[\mathrm{Pls}(B^*_{i_l i_m}) - \mathrm{Bel}(B^*_{i_l i_m})] - [\mathrm{Pls}(B^*_{i_j i_k}) - \mathrm{Bel}(B^*_{i_j i_k})]}. \tag{13}$$

If $\rho_{jklm} < 0.5$ then $B^*_{i_l i_m}$ is more preferred than $B^*_{i_j i_k}$. The requirement that we must have $\rho_{jklm} < 0.5$ is equivalent to having the mid-point in the belief-plausibility interval of $B^*_{i_l i_m}$ is higher than that of $B^*_{i_j i_k}$.

This implies that we can obtain an exact order of all partial sequence of preferences (of measures of performance) using a standard sorting algorithm based on the belief-plausibility interval mid-points for each sequence.

It is obvious from the representation of $B^*_{i_1 i_n}$ (8) that there is of partial sequence of preferences with sequence length 1 ($n = 2$)

$$B^*_{i_1 i_2} = \{(B_{i_1} \succcurlyeq B_{i_2})\} \tag{14}$$

where $|\{(B_{i_1} \succcurlyeq B_{i_2})\}| = 1$, $B_{i_1} \cap B_{i_2} = \emptyset$ and $B_{i_1} \cup B_{i_2} = \{MOE_i\}_{i \in I}$, i.e., B_{i_j} and B_{i_k} are exclusive and exhaustive.

When

$$B_{i_1} = \{MOE_i\}_{i \in I} \tag{15}$$

we have

$$\text{Bel}(B^*_{i_1 i_2}) = \text{Pls}(B^*_{i_1 i_2}) = 1 . \tag{16}$$

This makes it necessary to put a value on the information content of $B^*_{i_1 i_n}$ that is valued against the belief and plausibility of the partial sequence of preferences (of measures of effectiveness; MOEs), otherwise we will always prefer a fully nonspecific proposition with belief of 1 but with no information value (i.e., a vacuous belief function).

4 A Decision Maker's Value of Preferences

The value to a decision maker of a partial sequence of preferences $B^*_{i_1 i_n}$ (8) is obviously less than that of a complete sequence of preferences $B_{i_1 i_n}$ (7). As the sequence of preference is intended to be used for weight assignment for the different MOEs, where the weights assigned abide by the preference order, it is not possible to say which weight should be higher of MOE_{i_j} and MOE_{i_k} if they belong to the same subset, e.g., if

$$B^*_{i_1 i_n} = \{(B_{i_1} \succcurlyeq B_{i_2}) \cap (B_{i_2} \succcurlyeq B_{i_3})\} , \tag{17}$$

where

$$(B_{i_1} \succcurlyeq B_{i_2}) = (\{MOE_1, MOE_2\} \succcurlyeq \{MOE_3\})$$
$$(B_{i_2} \succcurlyeq B_{i_3}) = (\{MOE_3\} \succcurlyeq \{MOE_4\}) \tag{18}$$

we can only state that we must have $\{w_1, w_2\} \geq w_3 \geq w_4$ when weighing the preferences in MCDM, but we cannot say anything regarding the relative values of w_1 and w_2.

Finding the best partial sequence of preferences (of measures of effectiveness) becomes a balance between finding sequences with high belief-plausibility and high information value [13]. A measure that calculates a type of information value is the *aggregated uncertainty* (*AU*). The functional *AU* was independently discovered by several authors about the same time [2, 7, 10]. In general, *AU* is defined as

$$AU(\text{Bel}) = \max_{\{p_x\}_{x \in \Theta}} \left\{ -\sum_{x \in \Theta} p(x) \log_2 p(x) \right\} \tag{19}$$

where $\{p_x\}_{x \in \Theta}$ is the set of all probability distributions such that $p_x \in [0, 1]$ for all $x \in \Theta$.

Abellán et al. [1] suggested that AU could be disaggregated in separate measures of nonspecificity and scattering that generalize Hartley information [8] and Shannon entropy [18], respectively, for any mass function, i.e., $AU(m) = I(m) + GS(m)$. Dubois and Prade [5] defined such a measure of nonspecificity as

$$I(m) = \sum_{A \in F} m(A) \log_2 |A| \tag{20}$$

where $F \subseteq 2^{\Theta}$ is the set of focal elements.

The problem studied in this paper is a special case. We have a partial sequence of preferences where each set of preferences $(B_{i_j} \succcurlyeq B_{i_k})$ in the sequence corresponds to a mass function with one, usually nonspecific, focal element A with mass 1 and cardinality greater or equal than 1. Thus, with $m(A) = 1$ we have no scattering of information (i.e., $GS(m) = 0$) and AU specializes to $I(m)$ where the nonspecificity (20) simplifies further to the traditional Hartley function [8]

$$H(m) = \log_2 |A| , \tag{21}$$

as $m(A) = 1$, for each set of preferences in the sequence of $B^*_{i_1 i_n}$.

Using the problem representation of $B^*_{i_1 i_n}$ (8) the joint Hartley information of an entire sequence of multiple preference relations is formulated as

$$\begin{aligned} H(B^*_{i_1 i_n}) &= \log_2 \prod_{(B_{i_j} \succcurlyeq B_{i_k}) \in \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} \mid \cap \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} = B^*_{i_1 i_n}} |(B_{i_j} \succcurlyeq B_{i_k})| \\ &= \log_2 \prod_{(B_{i_j} \succcurlyeq B_{i_k}) \in \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} \mid \cap \{B_{i_l} \succcurlyeq B_{i_m}\}_{lm} = B^*_{i_1 i_n}} |B_{i_j}||B_{i_k}| \\ &= \log_2 \prod_{j=1}^{n} |B_{i_j}| = \sum_{j=1}^{n} \log_2 |B_{i_j}| , \end{aligned} \tag{22}$$

where the first equality use the definition of the Hartley function for multiple variables, the second equality use the fact that subsets of measures can be simplified to a set of preferences among single measures (2) [20].

Furthermore, we have

$$0 \le H(B^*_{i_1 i_n}) < |\{MOE_i\}_{i \in I}| \frac{\log_2 e}{e} \tag{23}$$

where the upper limit is reached when the number of preference subsets in the sequence is $n = \frac{1}{e} |\{MOE_i\}_{i \in I}|$, and the number of preference relations in each subset $|B_{i_j}| = \log_2 e, \forall j$ (ignoring that $n, |B_{i_j}| \in \mathbb{Z}^+$).

Note, that the best information value for the decision maker is when $H(B^*_{i_1 i_n})$ is minimized, i.e., when the sequence of preference is as specific as possible with one preference relation per subset (7). The only reason to prefer a partial sequence of preferences before a complete sequence is if its belief-plausibility is higher.

5 The Decision Maker's Choice of Preference Order

The utility U for a decision maker of knowing a sequence of preferences $B^*_{i_1 i_n}$ is a trade-off between finding a sequence of preferences that on the on hand maximize the belief and plausibility, and on the other hand maximize the value of the information itself for the decision maker.

A function that tries to achieve both task simultaneously by calculating the utility of $B^*_{i_1 i_n}$ is the product of the belief-plausibility midpoint, i.e., ρ = ½ (13), with a function of the Hartley function of $B^*_{i_1 i_n}$.

We define

$$U(B^*_{i_1 i_n}) = \frac{1}{2}\left[\mathrm{Bel}(B^*_{i_1 i_n}) + \mathrm{Pls}(B^*_{i_1 i_n})\right]\left[1 - \frac{H(B^*_{i_1 i_n})}{\left|\{MOE_i\}_{i \in I}\right|\frac{\log_2 e}{e}}\right], \tag{24}$$

where both terms on the right hand side of the equality belong to [0, 1]. Thus, the utility $U(B^*_{i_1 i_n}) \in [0, 1]$ and will serve as the basis for comparing different alternative partial sequences of preferences (of measures of performance; MOEs).

All partial sequences of preferences $B^*_{i_1 i_n}$ are evaluated based on their utility $U(B^*_{i_1 i_n})$. The partial sequence with highest utility is considered the best sequence and is the partial preference order that will be used in MCDM. Although (24) is exponential in the number of MOEs, the number of measures in the MCDM is usually not very large which makes this a calculation with low computational cost. In a previous paper [15] we developed a method for assigning weights by a Monte Carlo approach to the set of all MOEs for multiple criteria evaluation. When we have a partial sequence of preferences, e.g.,

$$B^*_{i_1 i_4} = \{(B_{i_1} \succcurlyeq B_{i_2}) \cap (B_{i_2} \succcurlyeq B_{i_3})\}, \tag{25}$$

where

$$(B_{i_1} \succcurlyeq B_{i_2}) = (MOE_1 \succcurlyeq \{MOE_2, MOE_3\})$$
$$(B_{i_2} \succcurlyeq B_{i_3}) = (\{MOE_2, MOE_3\} \succcurlyeq MOE_4) \tag{26}$$

we may assign any weight to the MOEs that abide by the constraints $1 \ge w_1 \ge \{w_2, w_3\} \ge w_4 \ge 0$ where w_i is the weight of MOE_i and there is no constraint between w_2 and w_3.

Other authors have considered different approaches to weight assignment. Huang et al. [9] consider the assignment of weights to criteria based on the consistency and similarity of the opinions from decision makers regarding these criteria. In addition it is also possible to let the decision makers themselves be weighted. Yue [21] suggest using the decision makers' experience regarding the topic under consideration as a basis for assigning weights. A third approach, is to let each decision maker use a weighting of his own as an expression of the importance placed on a pairwise comparison of two disjoint subsets of MOEs.

6 Conclusions

We show that it is possible to extend Utkin's methodology for complete ranking of all single preferences between different alternatives [20] in MCDM, to a new methodology that evaluates all partial rankings of all subsets of these measures. Both methods use the same pairwise comparisons of preference subsets assigned by experts. While Utkin's method use only plausibility for a complete ranking (of singletons), we show that this is not possible when extending the solution to incomplete ranking (of all possible subsets). Instead, it is necessary to calculate the utility by modelling the information value of an incomplete ranking and compare this, in a trade-off, against the belief-plausibility of the same incomplete ranking of all possible subsets of preferences (of measures of effectiveness; MOEs). Only then can we find the best partial ranking of preferences that combine high belief-plausibility with high information value to maximize utility for the decision maker.

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